Comment to Workshop on 2/10.

You are not supposed to assume that $a_n = \alpha - \frac{1}{n}$.

Recall: $\sup A = \alpha$

$\Leftrightarrow$ (i) $\forall a \in A,\ a \le \alpha$.

(ii) $\forall \varepsilon > 0,\ \exists a \in A.\ \alpha - \varepsilon < a \le \alpha$.

It should never be mistaken as equality

YOU HAVE ABSOLUTELY NO CONTROL OVER THE EXACT FORMULA FOR a.

All you know is that a lies somewhere in $(\alpha-\varepsilon, \alpha]$

Example: $A = \{ 1 - \frac{1}{e^n} : n = 1, 2, \cdots \}$. $\sup A = 1$

Then for $\varepsilon = \frac{1}{n}$, there exists some $1 - \frac{1}{e^{k_n}} \in (1 - \frac{1}{n}, 1]$

32(e): $\lim_{n\to\infty} n \cdot \left(\sqrt{4 - \frac{1}{n}} - 2 \right) = -\frac{1}{4}$.

$$n \cdot \left(\sqrt{4 - \frac{1}{n}} - 2 \right) = n \cdot \left(\sqrt{4 - \frac{1}{n}} - 2 \right) \cdot \frac{\sqrt{4 - \frac{1}{n}} + 2}{\sqrt{4 - \frac{1}{n}} + 2}$$

$$= n \cdot \frac{\cancel{4} - \frac{1}{n} - \cancel{4}}{\sqrt{4 - \frac{1}{n}} + 2} = \frac{-1}{\sqrt{4 - \frac{1}{n}} + 2} \to -\frac{1}{4}$$

43. $0 \le a \le b$. Does the sequence $\{ (a^n + b^n)^{\frac{1}{n}} \}$ converge?

$a = 1,\ b = 4.$ $(1^2 + 4^2)^{\frac{1}{2}} \approx 4.123.$

$(1^{20} + 4^{20})^{\frac{1}{20}} \approx 4.000$

$(1^{100} + 4^{100})^{\frac{1}{100}} \approx 4.000$

Guess: $n \to \infty$, $(a^n+b^n)^{\frac{1}{n}} \longrightarrow b$

Note that $a \geq 0 \Rightarrow a^n + b^n \geq b^n \Rightarrow (a^n+b^n)^{\frac{1}{n}} \geq b$

Want $(a^n+b^n)^{\frac{1}{n}} \leq$ something related to b.

Note that $a \leq b \Rightarrow a^n \leq b^n \Rightarrow a^n + b^n \leq b^n + b^n = 2 \cdot b^n$.

$\Rightarrow (a^n+b^n)^{\frac{1}{n}} \leq (2b^n)^{\frac{1}{n}} = 2^{\frac{1}{n}} \cdot b$.

So we have $b \leq (a^n+b^n)^{\frac{1}{n}} \leq 2^{\frac{1}{n}} \cdot b$.

Fact: $2^{\frac{1}{n}} \to 1$ as $n \to \infty$.

Pf of the fact: $x_n = 2^{\frac{1}{n}} - 1$. Want: $x_n \to 0$, $n \to \infty$.

$$(x_n+1)^n = 2$$

Recall: $(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{k} a^{n-k} b^k + \cdots + b^n$

If $a, b \geq 0$ $\quad \geq a^n + n \cdot a^{n-1} b$

$$\Rightarrow (1+x_n)^n = 2 \geq 1 + n \cdot x_n \Rightarrow x_n \leq \frac{1}{n}$$

Note that $x_n \geq 0$ obviously, so $0 \leq x_n \leq \frac{1}{n} \Rightarrow x_n \to 0$

With this fact, $2^{\frac{1}{n}} \cdot b \to b$ as $n \to \infty$.

Squeeze lemma $\Rightarrow \lim_{n\to\infty} (a^n+b^n)^{\frac{1}{n}} = b$.

Exercise: Prove that $(a_1^n + a_2^n + \cdots + a_k^n)^{\frac{1}{n}} \longrightarrow \max\{a_1, a_2, \cdots, a_k\}$.

Hint: Formulate a similar estimate & prove $k^{\frac{1}{n}} \to 1$ as $n \to \infty$

This can further generalize to. $\left[\int_a^b (f(x))^n dx\right]^{\frac{1}{n}} \longrightarrow \max_{a \leq x \leq b} f(x)$

for $f(x)$ nonnegative continuous on $[a,b]$

45. $x \in \mathbb{R} \Rightarrow \exists$ sequence of rational numbers converging to x.

Density Thm: $\forall\, a < b \in \mathbb{R}$, $\exists\, r \in \mathbb{Q}$, $a < r < b$.

With this theorem: $\forall\, x \in \mathbb{R}$, $\forall\, n \in \mathbb{Z}_+$, $\exists\, r_n \in \mathbb{Q}$,

s.t. $x - \frac{1}{n} < r_n < x$.

i.e. for each n, we have $r_n \in \mathbb{Q}$, s.t. $x - \frac{1}{n} < r_n < x$.

The sequence $\{r_n\}_{n=1}^{\infty}$ converges to x by Squeeze Lemma.

Proof of Density Theorem:

By Archimedean property, for the positive number $\frac{y-x}{2}$,

there exists $n \in \mathbb{Z}_+$, s.t. $\frac{1}{n} < \frac{y-x}{2}$.

Set m to be the largest integer s.t. $\frac{m}{n} < y$.

Why can I make such a choice?

b/c $\{m \in \mathbb{Z}_+ : \frac{m}{n} < y\}$ is bounded above by the integer

M that is larger than ny.

Claim: $\frac{m}{n}$ lies in (x, y).

Suppose $\frac{m}{n} \leq x$, then by $\frac{1}{n} < \frac{y-x}{2}$, we know

$$\frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} < x + \frac{y-x}{2} = \frac{x+y}{2} < y.$$

i.e. $m+1$ is larger than m, making $\frac{m+1}{n} < y$

Contradicting the choice of m.

$\Rightarrow x < \frac{m}{n} < y$.

More Comments to Workshop 2/9.

1. $\alpha \in \mathbb{R}$, $\phi \neq E \subseteq \mathbb{R}$, $\alpha E = \{\alpha x : x \in E\}$

$\sup \alpha E = ?$

<u>Ans</u>: if $\alpha \geq 0$, $\sup \alpha E = \alpha \cdot \sup E$

if $\alpha < 0$, $\sup \alpha E = \alpha \cdot \inf E$

Hint: (1) Consider one case, $\alpha > 0$. The other is easy once this is clear.

(2) $\forall y \in \alpha E$, $y \leq \alpha \cdot \sup E$. b/c $\forall y \in \alpha E, \exists x \in E, y = \alpha x$.

(3). $\forall \varepsilon > 0$, $\exists y \in \alpha E$, s.t. $\alpha \sup E - \varepsilon < y$.

(Use the facts $y = \alpha x$, some $x \in E$

and $\exists x \in E$, $\sup E - \frac{\varepsilon}{\alpha} < x$.)

3. $\{a_n\}$, $\{b_n\}$ convergent seq. of real numbers,

$a_n < b_n$, $\forall n \in \mathbb{Z}_+$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Hint: (1) First consider the case $a_n = 0$, $b_n > 0$.

The general case can be reduced to this case by setting $c_n = b_n - a_n$ and consider $\{c_n\}$, $\{0\}$.

(2) $(c_n > 0, \forall n \in \mathbb{N}) \wedge (c_n \to c \text{ as } n \to \infty)$

Argue that c cannot be negative.

If c were negative, Pick $0 < \varepsilon < \frac{1}{2}|c|$

$\exists N \in \mathbb{Z}_+$, $\forall n > N$, $c_n < c + \varepsilon < 0$.

Contradicting $c_n > 0$, $\forall n$.